Gauss Quadrature Rules for the Evaluation of $2\pi^{-1/2} \int_0^\infty \exp(-x^2) f(x) dx$

By David Galant

Abstract. Gauss quadrature rules for evaluating integrals of the form

$$2\pi^{-1/2} - \int_0^\infty \exp(-x^2) f(x) dx$$

have been calculated to 20S for one to twenty nodes. The coefficients for the three-term recurrence relation of the first twenty orthogonal polynomials associated with the weight function $\exp(-x^2)$ on the interval $[0, \infty)$ are also tabulated to 20S.

National Aeronautics and Space Administration
Ames Research Center
Moffett Field, California 94035

SECOND EDITORIAL REMARK (Added in proof). As the journal went to press, another article was received on this subject. We reprint the abstract of this article above and reproduce the extensive tables in the microfiche section of this issue. A review of the tables appears in the review section of this issue, RMT 42, p. 676.

Received March 3, 1969.

the eigenvalues and the first component of the orthonormal eigenvectors of the associated n x n tridiagonal matrix eigensystem as suggested in (2). The computation was carried out using 30S arithmetic.

Each rule was checked (before rounding to 20S) by generating all the odd moments for which the given rule was exact. In no case did the relative error exceed 10^{-27}

In the notation of the tables, a(n) denotes a * 10^{n}.

NASA Ames Research Center
Moffett Field, California 94035

References:

1. P. Henrici, "The Quotient-Difference Algorithm," Nat. Bur. Standards Appl. Math. Series, 49, 23-46 (1958).

2. G. H. Golub and J. Welsch, "Calculation of Gauss Quadrature Rules," Tech. Rep. No. CS81, Computer Science Dept., Stanford University.

Table I - Recurrence relation parameters for the orthogonal polynomials associated with $\exp(-x^2)$ on $[0, \infty)$.

j	b_j	g_j
1	0.5641895835 4775628695	0.0
2	0.9884253928 4680028549	0.1816901138 1620932846
3	0.1285967619 3639399603(1)	0.3413251289 5943919856
4	0.1524720844 0801153035(1)	0.5049621529 8800163194
5	0.1730192274 3094392568(1)	0.6702641946 3961908568
6	0.1913499843 1431025707(1)	0.8361704992 8031101555
7	0.2080620336 4008332248(1)	0.1002347851 0110108422(1)
8	0.2235228380 5046391487(1)	0.1168671164 7442727438(1)
9	0.2379782443 5046374209(1)	0.1335082922 2423353580(1)
10	0.2516025643 4438664098(1)	0.1501552599 3447618439(1)
11	0.2645247925 0569531803(1)	0.1668062362 1881161688(1)
12	0.2768435953 5042559069(1)	0.1834601052 7937676420(1)
13	0.2886364594 0326945693(1)	0.2001161318 5512137843(1)
14	0.2999655653 3536035387(1)	0.2167738111 7632644853(1)
15	0.3108817175 9249201517(1)	0.2334327849 5405013980(1)
16	0.3214270636 0711282274(1)	0.2500927917 1337026700(1)
17	0.3316370297 0830873659(1)	0.2667536360 9572020883(1)
18	0.3415417332 4133389445(1)	0.2834151691 6678327579(1)
19	0.3511670344 6156295154(1)	0.3000772753 7827190276(1)
20	0.3605353345 9055664303(1)	0.3167398636 9644268118(1)

Table II - Gauss quadrature rules for weight function $2/\pi^{\frac{1}{2}} \exp(-x^2)$ on $[0, \infty)$.

n	t_{jn}	w_{jn}
1	0.5641895835 4775628695	0.1000000000 0000000000(1)
2	0.3001939310 6083942187	0.7227597822 7263103940
	0.1252421045 3337171506(1)	0.2772402177 2736896060
3	0.1905541497 9819201556	0.5032907008 9897032872
	0.8482518675 4457669415	0.4473665328 9538306522
	0.1799776578 4157278230(1)	0.4934276620 5646606060(-1)
4	0.1337764469 9606762716	0.3670651279 1938403954
	0.6243246901 8718997107	0.4751684808 4583509379
	0.1342537825 6449922837(1)	0.1505737374 0856051721
	0.2262664477 0103619543(1)	0.7192653826 2203494523(-2)
5	0.1002421519 6821559632	0.2802963269 2725535375
	0.4828139660 4620070000	0.4426982022 1492815551
	0.1060949821 5257170812(1)	0.2385598846 1199914147
	0.1779729418 5202613018(1)	0.3751483891 5956856543(-1)
	0.2669760356 0876564137(1)	0.9307473298 6049273527(-3)
6	0.7860065941 3097918731(-1)	0.2221210728 7084638347
	0.3867394102 7063056412	0.3939783271 0512406020
	0.8664294716 8204387045	0.2902862835 6443996659
	0.1465698049 6635163067(1)	0.8577164087 5356964369(-1)
	0.2172707796 9389990015(1)	0.7731562231 2371504981(-2)
	0.3036820169 3228660022(1)	0.1111133529 9547486954(-3)
7	0.6371648460 6700796301(-1)	0.1812289387 0236205840
	0.3181920188 8861856438	0.3456448899 9429334201
	0.7241989692 5837302559	0.3108990867 5945889277
	0.1238035599 2150891549(1)	0.1361165968 5148496350
	0.1838528220 2709466310(1)	0.2470279983 0441068754(-1)
	0.2531488151 3276762003(1)	0.1395180729 4490470852(-2)
	0.3373456430 1246825160(1)	0.1250713251 0607486670(-4)
8	0.5297864393 1851128351(-1)	0.1513260143 6685698823
	0.2673983721 6776535132	0.3027788332 3794690538
	0.6163028841 8239990291	0.3113800653 7935190977
	0.1064246312 1162239988(1)	0.1776613620 0200952113
	0.1588855862 2700552793(1)	0.5056730802 2793567144(-1)
	0.2183921153 0958585978(1)	0.6057066877 4060864730(-2)
	0.2863133883 7080748824(1)	0.2280044121 0830955151(-3)
	0.3686007162 7243968973(1)	0.1345701526 7123219919(-5)

0.4493903080 1190497898(-1)
0.2286053055 6052255630
0.5321958443 3162262442
0.9272807453 3804910040
0.1392923855 1958459395(1)
0.1918843099 1973946155(1)
0.2506247834 0056995677(1)
0.3172692133 4811982052(1)
0.3978898869 7897403521(1)

0.3873852432 5699388059(-1)
0.1982333040 1294882169
0.4652011118 1450692293
0.8168618855 9190731519
0.1234541324 0277401312(1)
0.1706798149 6886490935(1)
0.2229940088 9244399308(1)
0.2809103746 8982532118(1)
0.3463872419 4953725258(1)
0.4255361806 3656125281(1)

0.3383932123 1774461827(-1)
0.1739557277 1023629922
0.4108738409 7238739204
0.7262717842 5989706112
0.1103863246 4649073238(1)
0.1532295034 5753714699(1)
0.2005782902 4681364849(1)
0.2524352141 5192080816(1)
0.3095351709 8692246667(1)
0.3739478609 9435776265(1)
0.4517835967 1873621815(1)

0.2988970076 9664384711(-1)
0.1542048782 6582584545
0.3661439629 7431241199
0.6508810158 4520456856
0.9943668698 8079202202
0.1385891203 6495647085(1)
0.1818848608 4282318609(1)
0.2290842738 6728545885(1)
0.2804096793 3936235417(1)
0.3367270704 1629265695(1)
0.4001683475 6734815137(1)
0.4768216287 9898575406(1)

0.1287356172 1657825568
0.2662306734 8482766599
0.3006289540 2793152046
0.2067773770 3359779493
0.8050313893 3913524765(-1)
0.1577634126 6566217496(-1)
0.1313267163 7017026458(-2)
0.3449119024 9240661999(-4)
0.1396826340 7737384454(-6)

0.1112041337 1404575759
0.2354679829 9032236261
0.2844098742 2601291571
0.2241912701 2751909042
0.1096766753 1710117577
0.3049378850 8830146628(-1)
0.4293087372 1310086933(-2)
0.2582704682 8707387472(-3)
0.4903196544 4284938067(-5)
0.1407920604 0211476570(-7)

0.9728964789 7768085724(-1)
0.2096159726 3536819252
0.2661012855 2422020002
0.2322772203 4842853876
0.1349329017 0048314518
0.4868316042 9522230877(-1)
0.1000607140 2569379744(-1)
0.1046167576 8177480764(-2)
0.4690890219 4695234683(-4)
0.6621979427 7287600705(-6)
0.1384685010 9884553656(-8)

0.8663456361 2576148233(-1)
0.1878142765 6854293807
0.2475606324 1547565769
0.2335931156 3833750580
0.1548862473 6418114443
0.6827334242 6382639010(-1)
0.1867896526 3836157272(-1)
0.2918083071 1451102900(-2)
0.2327141448 1355949237(-3)
0.7973702518 5923995701(-5)
0.8565882230 0932757363(-7)
0.1333692463 7714915033(-9)

0.2665112662 2384696521(-1)	0.7678212937 0637794628(-1)
0.1378918554 6993907004	0.1693214315 6221339135
0.3288286751 5834439096	0.2297454905 1048646699
0.5873785314 7370648427	0.2303071023 1970876799
0.9014808845 3539248990	0.1693914097 3901905648
0.1261296502 5823782963(1)	0.8739706427 1906645857(-1)
0.1660037131 9054402844(1)	0.2988982389 10061[illegible]631(-1)
0.2094109004 1874870152(1)	0.6345361611 7986251951(-2)
0.2563207025 2546767749(1)	0.7709551128 9931850238(-3)
0.3070912341 0296412974(1)	0.4793956344 6998047500(-4)
0.3626692011 8025500582(1)	0.1281354674 5101377275(-5)
0.4252207400 4793180790(1)	0.1067958576 6623414797(-7)
0.5008008343 2377746083(1)	0.1261645956 4490804440(-10)
0.2395678924 4107422060(-1)	0.6906919600 5674653327(-1)
0.1242403441 0991384415	0.1535295923 9983336814
0.2973385689 1648150747	0.2131020806 5001204061
0.5333292144 4328790677	0.2240710850 5718832614
0.8218731891 1693545749	0.1789805011 2246892445
0.1154067073 8744130445(1)	0.1047325263 2882371627
0.1523274791 4457242977(1)	0.4280127268 0319492560(-1)
0.1925338210 4771043148(1)	0.1157309856 2342732081(-1)
0.2358600766 6447748511(1)	0.1943939936 2985752510(-2)
0.2824093754 8449712055(1)	0.1872616914 3473232028(-3)
0.3326269358 7051412318(1)	0.9248075996 1311771828(-5)
0.3875104990 9875764001(1)	0.1961987628 4811732374(-6)
0.4492438071 6069161690(1)	0.1289669590 8969940515(-8)
0.5238431362 6752894117(1)	0.1174868656 1115491205(-11)
0.2168694269 8851218790(-1)	0.6256112594 5691116175(-1)
0.1126841969 5264204112	0.1399502905 3336338142
0.2704926203 8899756048	0.1977946233 8515952389
0.4869022892 3427358813	0.2160714452 3911641334
0.7530435742 3051871664	0.1844604410 0209495141
0.1060930872 0058329732(1)	0.1195378489 8146331891
0.1404254809 3705782357(1)	0.5644946977 6633406212(-1)
0.1778646218 5204429253(1)	0.1855391382 5419030574(-1)
0.2181707962 8455131994(1)	0.4031932105 0231404112(-2)
0.2613060672 5135550014(1)	0.5448908497 4579190225(-3)
0.3074617939 4974992012(1)	0.4230395636 7090962760(-4)
0.3571407977 4673486669(1)	0.1685444134 4339021043(-5)
0.4113735918 4559928906(1)	0.2880423182 8347559366(-7)
0.4723512894 9706533468(1)	0.1514486736 4858911412(-9)
0.5460488773 8648576500(1)	0.1078989046 3961630649(-12)

16	0 1975365846 0077267373(-1)	0.5701094219 8089655377(-1)
	0.1028022452 3791745096	0.1281935288 0314529073
	0 2473976694 5245507829	0.1838369921 2401078939
	0.4466962259 6168318716	0.2071284406 3326916437
	0.6930737203 0199955070	0 1866775122 7609121836
	0.9794041703 3072990638	0.1315379697 9701156992
	0.1299789321 2770360432(1)	0.6995916544 4541193556(-1)
	0 1649854240 3974342978(1)	0.2699050201 6696662867(-1)
	0.2026808152 1688666536(1)	0.7232813898 9632007982(-2)
	0.2429450491 6021432280(1)	0.1281494928 7492829666(-2)
	0.2858266528 5432659699(1)	0.1413703620 7239568745(-3)
	0 3315769275 0386984909(1)	0.8971173738 1040264772(-5)
	0.3807377116 7558980631(1)	0.2922510640 3666018945(-6)
	0.4343606345 4701723830(1)	0.4075196810 2274754397(-8)
	0.4946377204 0483858838(1)	0.1735083726 3314539271(-10)
	0.5675017934 0419219889(1)	0.9787791595 3220708497(-14)
17	0.1809108332 9126178186(-1)	0 5223291792 1440432179(-1)
	0.9427561332 9673095305(-1)	0.1179478804 2688349094
	0.2273676791 4088763098	0 1711670450 1068589502
	0.4116215432 1891186465	0.1977930462 4791333116
	0 6404653042 5866726517	0.1864015962 6844754415
	0.9075572451 6035634142	0.1407753580 3576831848
	0.1207442505 3077741014(1)	0.8265382315 1317036021(-1)
	0.1535860903 7651606022(1)	0.3643433350 3568424560(-1)
	0.1889836898 6846547098(1)	0.1161346196 7377168536(-1)
	0.2267684364 4391007930(1)	0.2568626187 0892569732(-2)
	0.2669032183 7933620079(1)	0.3757930991 4725824064(-3)
	0.3094965841 5622420766(1)	0 3426972003 8221515035(-4)
	0.3548415607 5716564391(1)	0.1799406388 1997438618(-5)
	0.4035068908 9905157132(1)	0.4849405814 0215972837(-7)
	0.4565576525 1009686654(1)	0 5579318102 7896244049(-9)
	0.5161826321 6192122623(1)	0.1944493879 1569422779(-11)
	0.5882726076 8995030625(1)	0.8781055992 1142017935(-15)
18	0.1664903222 0237232745(-1)	0.4808504544 1007498888(-1)
	0.8685906210 8408664468(-1)	0.1089637629 2187228317
	0.2098683481 3036414202	0 1596880005 9649859963
	0.3808200110 6888910098	0.1884255173 4359558101
	0.5940307207 5356708341	0.1842823686 4806288545
	0.8438630216 3542019276	0.1474785555 9844547455
	0.1125307376 3927296483(1)	0.9408225617 1891219353(-1)
	0.1434287710 4861852429(1)	0.4638905609 6032825481(-1)
	0.1767773935 6449368191(1)	0.1710358098 1484310170(-1)
	0.2123798695 1218886854(1)	0.4551056779 7454408917(-2)
	0.2501459458 7764926247(1)	0.8400040238 2653842704(-3)
	0.2900972235 6389061183(1)	0.1026219016 1809509873(-3)
	0.3323848568 9420072355(1)	0.7822185934 2304052880(-5)
	0.3773315134 3500571733(1)	0.3434982296 1500326133(-6)
	0.4255247112 6614397437(1)	0.7737366209 9295816995(-8)

0.4780378060 9304287768(1)	0.7417546649 1860684200(-10)
0.5370536579 8197246259(1)	0.2136480496 5353282052(-12)
0.6084216863 9034286434(1)	0.7799561358 0008262012(-16)

0.1538866601 3620514249(-1)	0.4445712213 9580185806(-1)
0.8036131828 2454333471(-1)	0.1010400769 7245889607
0.1944783268 1045215569	0.1492910452 1719506924
0.3536075958 5607368200	0.1792541657 9602965044
0.5528168629 1266648041	0.1808435806 2033499497
0.7870949226 3679575029	0.1519653203 3676041354
0.1051862150 4521038946(1)	0.1039961380 6519113051
0.1343264000 4529911408(1)	0.5638949630 2018695483(-1)
0.1658296307 3212607302(1)	0.2352217445 5841402137(-1)
0.1994846803 7534498429(1)	0.7316810492 2038658392(-2)
0.2351673706 7672439453(1)	0.1640769304 1082762056(-2)
0.2728441537 0144176103(1)	0.2552426060 1123061350(-3)
0.3125790153 4521341823(1)	0.2629980403 5839018159(-4)
0.3545539599 1836546944(1)	0.1692054085 0994338216(-5)
0.3991130559 8020907422(1)	0.6273268468 6031491678(-7)
0.4468570028 6680091188(1)	0.1191833437 7905392682(-8)
0.4988638745 3062028587(1)	0.9604061899 5408109873(-11)
0.5573088617 5719778993(1)	0.2305810703 6674604093(-13)
0.6280010370 9431417431(1)	0.6865270985 2995670196(-17)

0.1427950969 9916748348(-1)	0.4126248095 3539569702(-1)
0.7463130039 2191370694(-1)	0.9401377007 2616711019(-1)
0.1808615630 5803827340	0.1398673127 4031909417
0.3294333560 6428831671	0.1704175033 1594238999
0.5160505430 6153028240	0.1764948260 0028369422
0.7362554575 8089084032	0.1545791839 7722746942
0.9858735750 3752715715	0.1123067291 1046149556
0.1261289016 1027661602(1)	0.6604828742 7063176807(-1)
0.1559579645 2096599804(1)	0.3061944925 8661697089(-1)
0.1878561919 3029803500(1)	0.1088791653 3243534701(-1)
0.2216794165 3876453743(1)	0.2879302149 6894664392(-2)
0.2573577820 8262785150(1)	0.5488969148 1505559356(-3)
0.2948969746 7872328552(1)	0.7261283758 3353935432(-4)
0.3343981379 8616392289(1)	0.6366757009 3782589326(-5)
0.3760599932 9107743141(1)	0.3487635540 6925263130(-6)
0.4202442600 4211361556(1)	0.1100893097 8300542498(-7)
0.4675608847 7944769882(1)	0.1778423417 4416991060(-9)
0.5190901686 9750004295(1)	0.1214074765 5433194082(-11)
0.5769985165 5677557811(1)	0.2448428821 0576554230(-14)
0.6470558387 0645765819(1)	0.5993073454 5939542643(-18)

TABLES AND PROGRAMS FOR DIRICHLET L-SERIES

BY

ROBERT SPIRA

Table of CONTENTS

DESCRIPTION OF TABLES

TABLE I. Basis for M(K), the group of residue prime to K, K = 1(1)200. PHI = $\varphi(K)$, R = the number of basis elements Bi of order Hi. 5 pp.

TABLE II. Characters mod K, K = 1(1)24. N = character number, A = the residue in M(k) corresponding to this character, i.e., if $\chi(Bj) = \exp(\beta_j 2\pi i/Hj)$, $A = B1^{\beta 1} B2^{\beta 2} \ldots$. H = order of A. R = 0 if the character is primitive, and otherwise is the resolution modulus. Under T is the letter R or C according to whether the character is real or complex. The values of the character are then given at the natural numbers 1,2,...,K. If $\chi(j) = 0$ the value given is 0. If $\chi(j) = \exp(J2\pi i/\varphi(K))$, the value given is J, $1 \leq J \leq \varphi(K)$. 11 pp.

TABLE III. Primitive character N's mod K, K = 1(1)100. For each K, there is given a list of the N's corresponding to primitive characters, first the real ones, where the letter under T is R, and then the complex ones where under T will be C. The respective numbers are found under the heading NO. 5pp.

TABLE IV. $L(0,\chi)$, $L'(0,\chi)$, for primitive characters mod K, 15D, K = 3(1)24. N is the character number of χ and $L(0,\chi)$ = ReL + iImL, $L'(0,\chi)$ = ReL' + iImL'. In some cases, exact values are given. Values are given only for the smaller N from a pair of conjugate characters. 3 pp.